Symmetric coinvariant algebras and local Weyl modules at a double point

Toshiro Kuwabara

Department of Mathematics, Graduate School of Science, Kyoto University,
Kyoto 606-8502, Japan
Email: toshiro@math.kyoto-u.ac.jp

Abstract

The symmetric coinvariant algebra $\mathbb{C}[x_1,\ldots,x_n]_{S_n}$ is the quotient algebra of the polynomial ring by the ideal generated by symmetric polynomials vanishing at the origin. It is known that the algebra is isomorphic to the regular representation of S_n.

Replacing $\mathbb{C}[x]$ with $A=\mathbb{C}[x,y]/(xy)$, we introduce another symmetric coinvariant algebra $A^{\otimes n}_{S_n}$ and determine its S_n-module structure. As an application, we determine the $\mathfrak{sl}_{r+1}$-module structure of the local Weyl module at a double point for $\mathfrak{sl}_{r+1}\otimes A$.

Keywords: symmetric groups, coinvariant algebras, infinite-dimensional Lie algebras, Weyl modules.

1 Introduction

The symmetric group S_n acts on the polynomial ring of n variables $\mathbb{C}[x_1,\ldots,x_n]$. Let $\mathbb{C}[x_1,\ldots,x_n]^{S_n}_+$ be the set of symmetric polynomials vanishing at the origin $x_1=\cdots=x_n=0$. For an algebra R and a subset S of R, let $\langle S\rangle_R$ be the ideal of R generated by S. The classical symmetric coinvariant algebra $\mathbb{C}[x_1,\ldots,x_n]_{S_n}$ is defined as the quotient algebra:

$$\mathbb{C}[x_1,\ldots,x_n]_{S_n}=\mathbb{C}[x_1,\ldots,x_n]/\langle\mathbb{C}[x_1,\ldots,x_n]^{S_n}_+\rangle_{\mathbb{C}[x_1,\ldots,x_n]}.$$

It is known that this is isomorphic to the regular representation of S_n as an S_n-module ([C]).

The symmetric group S_n acts diagonally on the polynomial ring of $2n$ variables $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$, i.e.

$$\sigma P(x_1,\ldots,x_n,y_1,\ldots,y_n)=P(x_{\sigma(1)},\ldots,x_{\sigma(n)},y_{\sigma(1)},\ldots,y_{\sigma(n)})$$

for $\sigma\in S_n$ and $P\in\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. Recently, Haiman defined the diagonal symmetric coinvariant algebra

$$\begin{aligned}&\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]_{S_n}=\\&\qquad\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]/\langle\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n}_+\rangle_{\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]}\end{aligned}$$

where $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n}_+$ is symmetric polynomials vanishing at the origin. He determined its S_n-module structure in the form

$$\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]_{S_n}\simeq\mathbb{C}PF_n\otimes L_{(1^n)}$$

where PF_n is the set of parking functions, functions from $\{1,\ldots,n\}$ to itself satisfying some condition, $\mathbb{C}PF_n$ is the vector space spanned by PF_n, and $L_{(1^n)}$

is the sign representation of S_n ([Ha]). In generally, for a partition λ we denote by L_λ the irreducible representation of S_n corresponding to λ.

Let M be an affine variety over $\mathbb{C}$ and let A be its coordinate ring. The symmetric coinvariant algebra $A^{\otimes n}_{S_n}$ is introduced by Feigin and Loktev in [FL]. The symmetric group S_n acts on $A^{\otimes n}$, the n-th tensor product of A. Fix a base point 0 on M. Let $(A^{\otimes n})^{S_n}_+$ be the set of symmetric elements vanishing at the point $(0, \ldots, 0)$. The symmetric coinvariant algebra is defined as

$$A^{\otimes n}_{S_n} = A^{\otimes n} / \langle (A^{\otimes n})^{S_n}_+ \rangle_{A^{\otimes n}}.$$

This representation is used for the study of the structure of $(\mathfrak{sl}_{r+1} \otimes A)$-module $W_M(\{0\}_\lambda)$ called the local Weyl module. Let $\mathfrak{sl}_{r+1} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the triangular decomposition of $\mathfrak{sl}_{r+1}$, and let $\lambda \in \mathfrak{h}^*$ be a dominant integrable weight. The local Weyl module $W_M(\{0\}_\lambda)$ is the maximal $\mathfrak{sl}_{r+1}$-integrable $(\mathfrak{sl}_{r+1} \otimes A)$-module generated by a cyclic vector v_0 with the following properties:

$$(\mathfrak{n}_+ \otimes P)v_0 = 0, \quad (h \otimes P)v_0 = \lambda(h)P(0)v_0 \quad \text{for all } P \in A,\ h \in \mathfrak{h}.$$

This definition was first given by Chari and Pressley in [CP] for $A = \mathbb{C}[x]$ and then generalized by Feigin and Loktev in [FL].

Let $V_{r+1} = \mathbb{C}^{r+1}$ be the vector representation of $\mathfrak{sl}_{r+1}$ and let ω_1 be the highest weight of V_{r+1}. In [FL], Feigin and Loktev show that there is an isomorphism of $\mathfrak{sl}_{r+1}$-modules :

$$W_M(\{0\}_{n\omega_1}) \simeq \left(V^{\otimes n}_{r+1} \otimes A^{\otimes n}_{S_n}\right)^{S_n}.$$

This isomorphism gives us the connection between the S_n-module structure of the symmetric coinvariant algebra and the $\mathfrak{sl}_{r+1}$-module structure of the local Weyl module.

In this paper, We consider the case of $A = \mathbb{C}[x, y]/(xy)$. In this case, the corresponding affine variety M has the double point 0. We consider the symmetric coinvariant algebra and the local Weyl module at the double point 0. Our main result is

Theorem 1. *We have the following isomorphism of S_n-modules*:

$$A^{\otimes n}_{S_n} \simeq \mathbb{C}[S_n] \oplus (n-1) \operatorname{Ind}^{S_n}_{S_2} L_{(1,1)}.$$

where $L_{(1,1)}$ is the sign representation of S_2.

As a corollary of Theorem 1, we determine the structure of the local Weyl module $W_M(\{0\}_{n\omega_1})$.

Proposition 2. *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$W_M(\{0\}_{n\omega_1}) \simeq V^{\otimes n}_{r+1} \oplus (n-1) \left(V^{\otimes n-2}_{r+1} \otimes \wedge^2 V_{r+1}\right)$$

as an $\mathfrak{sl}_{r+1}$-module.

Let us give a sketch of the proof of Theorem 1. We introduce a generalization of the symmetric coinvariant algebra $R^n_{i,j}$. Let $e_1, \ldots, e_n$ be the elementary symmetric polynomials of variables $x_1, \ldots, x_n$, and $f_1, \ldots, f_n$ these of y_1,

$\dots, y_n$. For $I = \{k_1, \dots, k_i\} \subset \{1, \dots, n\}$, let $x_I = x_{k_1} \dots x_{k_i}$ and let $y_I = y_{k_1} \dots y_{k_n}$. The algebra $R^n_{i,j}$ is defined as

$$R^n_{i,j} = A^{\otimes n} / \langle e_1, \dots, e_{i-1}, x_I \ (|I| = i), f_1, \dots, f_{j-1}, y_J \ (|J| = j) \rangle_{A^{\otimes n}}.$$

Clearly we have $R^n_{n,n} = A^{\otimes n}_{S_n}$. By using the same method as that in [GP], we can determine the S_n-module structure of $R^n_{i,j}$ for $i, j \geq 1$, $i + j \leq n + 1$:

$$R^n_{i,j} \simeq \operatorname{Ind}^{S_n}_{S_{n-i-j+2}} L_{(n-i-j+2)}. \tag{1}$$

Next, we introduce the decreasing filtration $\{F^p A^{\otimes n}\}_{0 \leq p \leq n}$ of $A^{\otimes n}$ given by

$$F^p A^{\otimes n} = \sum_{k_1 < \dots < k_p} y_{k_1} \dots y_{k_p} A^{\otimes n}.$$

Let $\{F^p A^{\otimes n}_{S_n}\}_{0 \leq p \leq n}$ be its induced filtration on $A^{\otimes n}_{S_n}$. For $1 \leq i \leq n-1$, We have the following exact sequence

$$0 \to \operatorname{gr}^i A^{\otimes n}_{S_n} \to R^n_{n-i,i+1} \to R^n_{n-i,i} \to 0, \tag{2}$$

where $\operatorname{gr} A^{\otimes n}_{S_n}$ is the graded module associated to $\{F^p A^{\otimes n}_{S_n}\}_{0 \leq p \leq n}$. By combining (1) and (2), we obtain Theorem 1.

The paper is organized as follows. In Section 2, we recall basic definitions and notations. In Section 3, the symmetric coinvariant algebra is defined. In Section 4, we introduce the generalization of the symmetric coinvariant algebra and prove (1). In Section 5, we prove (2) and then Theorem 1. In Section 6, we review the definition of the local Weyl module and determine its structure.

Acknowledgments.

This research is partially supported by Grant-in-Aid for JSPS Research Fellows No.16-1089. The author is deeply grateful to Masaki Kashiwara, Yoshihiro Takeyama and Boris Feigin for useful discussions, Sergai Loktev for comments about Section 6. The author also thanks his advisor Tetsuji Miwa for reading the manuscript and for his kind encouragement.

2 Preliminaries

In this section we review some definitions and notations in the representation theory of symmetric groups and symmetric polynomials.

Let S_n be the n-th symmetric group. For each partition λ of n, let L_λ be the irreducible representation of S_n corresponding to λ.

For a finite group G, we denote by $\mathbb{C}[G]$ its group ring. For a G-module L and a subgroup H of G, L^H is the subspace of H-invariants. For an H-module L, we denote the induced module of L by $\operatorname{Ind}^G_H L = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} L$.

The following lemma easily follows from the semi-simplicity of representations of S_n.

Lemma 3. *If an S_n-module L has a filtration invariant under the action of S_n, we have an isomorphism $L \simeq \operatorname{gr} L$ of S_n-modules where $\operatorname{gr} L$ is the graded module associated with the filtration of L.*

Let V be a vector space. We denote by $V^{\otimes n}$ the n-th tensor product of V, and by $S^n(V)$ the n-th symmetric tensor product. For a subset $B \subset V$, we denote by $\mathrm{span}_{\mathbb{C}} B$ the subspace spanned by B.

For a set of indeterminates S, $e_k(S)$ is the k-th elementary symmetric polynomial of variables S. We write $e_1, \ldots, e_n$ for $e_i = e_i(\{x_1, \ldots, x_n\})$, and $f_1, \ldots, f_n$ for $f_i = e_i(\{y_1, \ldots, y_n\})$. When the number of variables matters, we use notations $e_1^{(n)}, \ldots, e_n^{(n)}$ and $f_1^{(n)}, \ldots, f_n^{(n)}$.

For a set of indices $I = \{j_1, \ldots, j_i\} \subset \{1, \ldots, n\}$ with $j_1 < \cdots < j_i$, we define $x_I = x_{j_1} \ldots x_{j_i}$. We also define y_I similarly.

For an algebra R and a subset S of R, let $\langle S \rangle_R$ be the ideal of R generated by S.

3 The symmetric coinvariant algebra R_n

First we review the classical symmetric coinvariant algebra $\mathbb{C}[x_1, \ldots, x_n]_{S_n}$. The symmetric group S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$. Therefore we can think of the ring of symmetric polynomials $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$, and we set

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n}_{+} = \{P \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} \mid P(0, \ldots, 0) = 0\}.$$

Consider the quotient algebra

$$\mathbb{C}[x_1, \ldots, x_n]_{S_n} = \mathbb{C}[x_1, \ldots, x_n] / \langle \mathbb{C}[x_1, \ldots, x_n]^{S_n}_{+} \rangle_{\mathbb{C}[x_1, \ldots, x_n]}.$$

We call this algebra the symmetric coinvariant algebra according to [Hi]. It is a classical result that we have

$$\mathbb{C}[x_1, \ldots, x_n]_{S_n} \simeq \mathbb{C}[S_n]$$

as an S_n-module.

Next, let $A = \mathbb{C}[x, y]/(xy)$ and let $M = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ be the corresponding affine variety. For any $n \in \mathbb{N}$, S_n acts on $A^{\otimes n}$. Set

$$(A^{\otimes n})^{S_n}_{+} = \{P \in (A^{\otimes n})^{S_n} \mid P(0, \ldots, 0) = 0\}$$

and let J_n be the ideal of $A^{\otimes n}$ generated by $(A^{\otimes n})^{S_n}_{+}$, i.e.

$$J_n = \langle (A^{\otimes n})^{S_n}_{+} \rangle_{A^{\otimes n}}.$$

Definition 4. *The symmetric coinvariant algebra $R_n = A^{\otimes n}_{S_n}$ is*

$$R_n = A^{\otimes n}_{S_n} = A^{\otimes n} / J_n.$$

Let π be the projection

$$\pi : A^{\otimes n} \longrightarrow R_n.$$

In this paper, we study the S_n-module structure of R_n.

By a theorem of Weyl [W], the elements $\sum_{i=1}^n x_i^r y_i^s$ $(r,\ s \geq 0)$ generate $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$. In $A^{\otimes n}$, $\sum_{i=1}^n x_i^r y_i^s = 0$ for (r, s) such that $r \geq 1$ and $s \geq 1$. Therefore the ideal J_n is generated by the power sums $\sum_{i=1}^n x_i^r$ and $\sum_{i=1}^n y_i^r$ $(r \geq 1)$, or the elementary symmetric polynomials e_i, f_i $(1 \leq i \leq n)$.

4 Generalization of symmetric coinvariant algebra

In this section, we introduce a generalization of R_n and determine its S_n-structure.

Definition 5. *For $1 \leq i,\ j \leq n$, let $R^n_{i,j}$ be the quotient algebra of $A^{\otimes n}$ given by*

$$R^n_{i,j} = A^{\otimes n}/I^n_{i,j},$$

where

$$\begin{aligned} S^n_{i,j} &= \left\{ \begin{array}{ll} e_1, \ldots, e_{i-1}, x_I & (|I| = i) \\ f_1, \ldots, f_{j-1}, y_J & (|J| = j) \end{array} \right\} \subset A^{\otimes n}, \\ I^n_{i,j} &= \langle S^n_{i,j} \rangle_{A^{\otimes n}}. \end{aligned}$$

Let $\pi^n_{i,j}$ be the projection

$$\pi^n_{i,j} : A^{\otimes n} \longrightarrow R^n_{i,j}.$$

Clearly, R_n is equal to $R^n_{n,n}$. First we show the following variant of Newton identity for the elementary symmetric polynomials.

Lemma 6 (nonsymmetric Newton identity). *For $1 \leq i \leq n$, we have the following identity:*

$$x_n^i - x_n^{i-1} e_1^{(n)} + \cdots + (-1)^{i-1} x_n e_{i-1}^{(n)} + (-1)^i e_i^{(n)} = (-1)^i e_i^{(n-1)}. \tag{3}$$

Proof. Clearly, we have

$$(1 + x_n t)^{-1} \prod_{j=1}^{n} (1 + x_j t) = \prod_{j=1}^{n-1} (1 + x_j t). \tag{4}$$

For $1 \leq i \leq n$, the coefficient of t^i in (4) coincides with (3) (up to sign). □

The following lemma is easy to prove.

Lemma 7. *We have equalities $x_n^i = 0$ and $y_n^j = 0$ in $R^n_{i,j}$.*

Proof. By Lemma 6, we have

$$x_n^i - x_n^{i-1} e_1^{(n)} + \cdots + (-1)^{i-1} x_n e_{i-1}^{(n)} + (-1)^i e_i^{(n)} = (-1)^i e_i^{(n-1)}.$$

Since the elements $e_1^{(n)}$, …, $e_i^{(n)}$, and $e_i^{(n-1)}$ belong to $I^n_{i,j}$, we have $x_n^i \in I^n_{i,j}$. Similarly, we have $y_n^j \in I^n_{i,j}$. □

In the rest of this section, we determine the S_n-module structure of $R^n_{i,j}$ for $i + j \leq n + 1$. Our proof is a modification of that in [GP]. First, we introduce another S_n-module R_W.

For $i,\ j \geq 1$, $i + j \leq n + 1$, let a_1, …, $a_{i-1} \in \mathbb{C}^\times$ be distinct, and let b_1, …, $b_{j-1} \in \mathbb{C}^\times$ be also distinct. We set

$$z_0 = \left(\begin{pmatrix} a_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} a_{i-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ b_{j-1} \end{pmatrix}, \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{n-i-j+2} \right) \in M^n.$$

The symmetric group S_n acts on M^n. Let W be the S_n-orbit of z_0, then $\#W = \#(S_n/S_{n-i-j+2}) = n!/(n-i-j+2)!$.

We define

$$\begin{aligned} I_W &= \left\{P \in A^{\otimes n} | P(z) = 0 \quad (z \in W)\right\}, \\ R_W &= A^{\otimes n}/I_W. \end{aligned}$$

The algebra R_W is the coordinate ring of $W \simeq S_n/S_{n-i-j+2}$. Hence $R_W \simeq \mathrm{Ind}_{S_{n-i-j+2}}^{S_n} L_{(n-i-j+2)}$ as an S_n-module.

The algebra $A^{\otimes n}$ is graded with the homogeneous degree in x and y. We define the increasing filtration $\{G_p A^{\otimes n}\}_{p\geq 0}$ of $A^{\otimes n}$: $G_p A^{\otimes n}$ is the set of elements of $A^{\otimes n}$ whose homogeneous degree are less than p. We also define the filtration $\{G_p R_W\}_{p\geq 0}$ of quotient algebra R_W as its induced filtration.

Lemma 8. *Each element of $S^n_{i,j}$ is the leading homogeneous component of a polynomial in I_W*

Proof. For e_k $(1 \leq k \leq i-1)$ or f_l $(1 \leq l \leq j-1)$, the elements

$$e_k - e_k(a_1, \ldots, a_{i-1}, 0, \ldots, 0), \quad f_l - f_l(b_1, \ldots, b_{j-1}, 0, \ldots, 0)$$

belong to I_W and their leading homogeneous components are e_k or f_l.

The remaining generators x_I $(|I| = i)$ and y_J $(|J| = j)$ clearly belong to I_W. $\square$

From this lemma, we get the following surjective homomorphism of S_n-modules:

$$R^n_{i,j} \longrightarrow \mathrm{gr}\, R_W$$

where $\mathrm{gr}\, R_W$ is the graded algebra associated with the filtration $\{G_p R_W\}_{p\geq 0}$. Since the filtration of R_W is invariant by the action of S_n, Lemma 3 implies that $\mathrm{gr}\, R_W$ is isomorphic to R_W as an S_n-module. Thus we obtain the following proposition.

Proposition 9. *For i, $j \geq 1$ such that $i + j \leq n + 1$, there is a surjective homomorphism of S_n-modules*:

$$R^n_{i,j} \longrightarrow \mathrm{Ind}_{S_{n-i-j+2}}^{S_n} L_{(n-i-j+2)}.$$

Note that $\dim R^n_{i,j} \geq n!/(n-i-j+2)!$ by Proposition 9. Next, we show $\dim R^n_{i,j} \leq n!/(n-i-j+2)!$. First, we consider the case of $i + j = n + 1$.

We introduce the following filtration of $R^n_{i,j}$ for i, $j \geq 1$ such that $i+j = n+1$:

$$\begin{aligned} &0 = \langle y_n^j \rangle_{R^n_{i,j}} \subset \langle y_n^{j-1} \rangle_{R^n_{i,j}} \subset \cdots \subset \langle y_n^1 \rangle_{R^n_{i,j}} \\ &\subset \langle y_n, \mathcal{X}^{(n-1)}_{i-1} \rangle_{R^n_{i,j}} \subset \langle x_n^{i-2}, y_n, \mathcal{X}^{(n-1)}_{i-1} \rangle_{R^n_{i,j}} \cdots \subset \langle x_n^1, y_n, \mathcal{X}^{(n-1)}_{i-1} \rangle_{R^n_{i,j}} \subset R^n_{i,j} \end{aligned}$$

where $\mathcal{X}^{(n-1)}_{i-1} = \{x_I \mid I \subset \{1, \ldots, n-1\}, |I| = i-1\}$. Note that, since $x_n^{i-1} \equiv (-1)^{i-1} e^{(n-1)}_{i-1}$ by Lemma 6, $\langle y_n, \mathcal{X}^{(n-1)}_{i-1} \rangle_{R^n_{i,j}} = \langle y_n, x_n^{i-1}, \mathcal{X}^{(n-1)}_{i-1} \rangle_{R^n_{i,j}}$. From this

filtration, we have the following decomposition of $R^n_{i,j}$:

$$\begin{aligned} R^n_{i,j} \simeq R^n_{i,j}/\langle x_n, y_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}} \\ \oplus \bigoplus_{k=1}^{i-1} \langle x_n^k, y_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}/\langle x_n^{k+1}, y_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}} \\ \oplus \bigoplus_{k=1}^{j-1} \langle y_n^k\rangle_{R^n_{i,j}}/\langle y_n^{k+1}\rangle_{R^n_{i,j}}. \end{aligned} \tag{5}$$

Lemma 10. *For $i, j \geq 1$ such that $i+j = n+1$, $1 \leq k \leq i-1$ and $1 \leq k' \leq j-1$, we have the following surjective homomorphisms of S_{n-1}-modules*:

$$\begin{aligned} \varphi_0 &: R^{n-1}_{i-1,j} \to R^n_{i,j}/\langle y_n, x_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i-1,j} &\mapsto P + I^n_{i,j} + \langle y_n, x_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{A^{\otimes n}}, \\ \varphi_k &: R^{n-1}_{i-1,j} \to \langle y_n, x_n^k, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}/\langle y_n, x_n^{k+1}, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i-1,j} &\mapsto x_n^k P + I^n_{i,j} + \langle y_n, x_n^{k+1}, \mathcal{X}^{(n-1)}_{i-1}\rangle_{A^{\otimes n}}, \\ \varphi'_{k'} &: R^{n-1}_{i,j-1} \to \langle y_n^k\rangle_{R^n_{i,j}}/\langle y_n^{k+1}\rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i,j-1} &\mapsto y_n^k P + I^n_{i,j} + \langle y_n^{k+1}\rangle_{A^{\otimes n}}. \end{aligned}$$

Proof. First, since $e_l^{(n-1)} \equiv e_l^{(n)}$ modulo $\langle x_n\rangle_{A^{\otimes n}}$ and $f_l^{(n-1)} \equiv f_l^{(n)}$ modulo $\langle y_n\rangle_{A^{\otimes n}}$, we have $I^{n-1}_{i-1,j} \subset I^n_{i,j} + \langle y_n, x_n, \mathcal{X}^{(n-1)}_{i-1}\rangle_{A^{\otimes n}}$. Therefore, φ_0 is well-defined, and clearly it is surjective.

Next, we show that φ_k is well-defined for $1 \leq k \leq i-1$. Let $P \in A^{\otimes n-1} \subset A^{\otimes n}$, and assume P belongs to $I^{n-1}_{i-1,j}$. We have

$$\begin{aligned} P = P_1 e_1^{(n-1)} + \cdots + P_{i-2} e_{i-2}^{(n-1)} + \sum_{\substack{I \subset \{1,\ldots,n-1\} \\ |I| = i-1}} P_I x_I \\ + Q_1 f_1^{(n-1)} + \cdots + Q_{j-1} f_{j-1}^{(n-1)} + \sum_{\substack{J \subset \{1,\ldots,n-1\} \\ |J| = j}} Q_J y_J \end{aligned}$$

where $P_1, \ldots, P_{i-2}, P_I, Q_1, \ldots, Q_{j-1}, Q_J \in A^{\otimes n-1}$. Therefore, we have

$$\begin{aligned} x_n^k P = P_1 x_n^k e_1^{(n-1)} + \cdots + P_{i-2} x_n^k e_{i-2}^{(n-1)} + \sum_{\substack{I \subset \{1,\ldots,n-1\} \\ |I| = i-1}} x_n^k P_I x_I \\ + Q_1 x_n^k f_1^{(n-1)} + \cdots + Q_{j-1} x_n^k f_{j-1}^{(n-1)} + \sum_{\substack{J \subset \{1,\ldots,n-1\} \\ |J| = j}} x_n^k Q_J y_J. \end{aligned}$$

Since $x_n^k e_l^{(n-1)} \equiv x_n^k e_l^{(n)}$ modulo $\langle x_n^{k+1}\rangle_{A^{\otimes n}}$ and $x_n^k f_l^{(n-1)} = x_n^k f_l^{(n)}$, $x_n^k P$ belongs to $I^n_{i,j} + \langle y_n, x_n^{k+1}, \mathcal{X}^{(n-1)}_{i-1}\rangle_{A^{\otimes n}}$. Therefore, φ_k is well-defined.

For any element of $\langle y_n, x_n^k, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}/\langle y_n, x_n^{k+1}, \mathcal{X}^{(n-1)}_{i-1}\rangle_{R^n_{i,j}}$, we can choose its representative $x_n^k P$ where $P \in A^{\otimes n-1}$. Therefore φ_k is surjective.

Similarly, $\varphi'_{k'}$ is well-defined and surjective. $\square$

Proposition 11. *For $i,\ j \geq 1$ such that $i+j=n+1$, the S_n-module $R^n_{i,j}$ is isomorphic to the regular representation of S_n.*

Proof. By Proposition 9, we have the surjective homomorphism $R^n_{i,j} \twoheadrightarrow \mathbb{C}[S_n]$. We show $\dim R^n_{i,j} \leq n!$ by induction on n. First, consider the case of $n=1$. In this case, we have $i=j=1$, and this case is already proved. We may assume that $\dim R^{n-1}_{i',j'} = (n-1)!$ for $i',\ j' \geq 1$ such that $i'+j'=n$. Therefore, by (5) and Lemma 10, we have

$$\begin{aligned}\dim R^n_{i,j} &\leq \dim R^{n-1}_{i-1,j} + \sum_{k=1}^{i-1} \dim R^{n-1}_{i-1,j} + \sum_{k'=1}^{j-1} \dim R^{n-1}_{i,j-1} \\ &= n!.\end{aligned}$$

Hence, the induction completes. $\square$

Next, consider the case of $i+j \leq n$. We introduce the following filtration of $R^n_{i,j}$ for $i,\ j \geq 1$ such that $i+j \leq n$:

$$\begin{aligned}0 = \langle y_n^j \rangle_{R^n_{i,j}} \subset \langle y_n^{j-1} \rangle_{R^n_{i,j}} \subset \cdots \subset \langle y_n^1 \rangle_{R^n_{i,j}} \\ \subset \langle x_n^{i-1}, y_n \rangle_{R^n_{i,j}} \subset \langle x_n^{i-2}, y_n \rangle_{R^n_{i,j}} \subset \cdots \subset \langle x_n^1, y_n \rangle_{R^n_{i,j}} \subset R^n_{i,j}.\end{aligned}$$

From this filtration, we have the following decomposition of $R^n_{i,j}$:

$$\begin{aligned}R^n_{i,j} \simeq R^n_{i,j}/\langle x_n, y_n \rangle_{R^n_{i,j}} \oplus \bigoplus_{k=1}^{i-1} \langle x_n^k, y_n \rangle_{R^n_{i,j}} / \langle x_n^{k+1}, y_n \rangle_{R^n_{i,j}} \\ \oplus \bigoplus_{k=1}^{j-1} \langle y_n^k \rangle_{R^n_{i,j}} / \langle y_n^{k+1} \rangle_{R^n_{i,j}}. \quad (6)\end{aligned}$$

We can prove the following lemma similarly to Lemma 10.

Lemma 12. *For $i,\ j \geq 1$ such that $i+j \leq n$, $1 \leq k \leq i-1$ and $1 \leq k' \leq j-1$, we have the following surjective homomorphisms of S_{n-1}-modules*:

$$\begin{aligned}\varphi_0 : R^{n-1}_{i,j} &\to R^n_{i,j}/\langle y_n, x_n \rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i,j} &\mapsto P + I^n_{i,j} + \langle y_n, x_n \rangle_{A^{\otimes n}}, \\ \varphi_k : R^{n-1}_{i-1,j} &\to \langle y_n, x_n^k \rangle_{R^n_{i,j}} / \langle y_n, x_n^{k+1} \rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i-1,j} &\mapsto x_n^k P + I^n_{i,j} + \langle y_n, x_n^{k+1} \rangle_{A^{\otimes n}}, \\ \varphi'_{k'} : R^{n-1}_{i,j-1} &\to \langle y_n^k \rangle_{R^n_{i,j}} / \langle y_n^{k+1} \rangle_{R^n_{i,j}}, \\ P + I^{n-1}_{i,j-1} &\mapsto y_n^k P + I^n_{i,j} + \langle y_n^{k+1} \rangle_{A^{\otimes n}}.\end{aligned}$$

Proposition 13. *For $i,\ j \geq 1$ such that $i+j \leq n$, we have the following S_n-module isomorphism*:

$$R^n_{i,j} \simeq \mathrm{Ind}^{S_n}_{S_{n-i-j+2}} L_{(n-i-j+2)}.$$

Proof. The following proof is similar to one of Proposition 11. By Proposition 9, we have the surjective homomorphism $R^n_{i,j} \twoheadrightarrow \mathrm{Ind}^{S_n}_{S_{n-i-j+2}} L_{(n-i-j+2)}$. We show $\dim R^n_{i,j} \leq n!/(n-i-j+2)!$ by induction on n. First, consider the case of $n = 2$. In this case, we have $i = j = 1$, and this case is already proved. We may assume that $\dim R^{n-1}_{i',j'} = (n-1)!/(n-i'-j'+1)!$ for i', $j' \geq 1$ such that $i'+j' \leq n-1$. By Proposition 11, we have that $\dim R^n_{i,j} = n! = n!/(n-i-j+2)!$ for i, $j \geq 1$ such that $i + j = n + 1$. Therefore, by (6) and Lemma 12, we have

$$\begin{aligned}
\dim R^n_{i,j} &\leq \dim R^{n-1}_{i,j} + \sum_{k=1}^{i-1} \dim R^{n-1}_{i-1,j} + \sum_{k'=1}^{j-1} \dim R^{n-1}_{i,j-1} \\
&= \frac{(n-1)!}{(n-i-j+1)!} + (i-1)\frac{(n-1)!}{(n-i-j+2)!} + (j-1)\frac{(n-1)!}{(n-i-j+2)!} \\
&= \frac{n!}{(n-i-j+2)!}.
\end{aligned}$$

Therefore, the induction completes. □

5 The structure of R_n

In this section, we determine the S_n-module structure of R_n.

We define a decreasing filtration $\{F^i A^{\otimes n}\}_{0 \leq i \leq n}$ of $A^{\otimes n}$ where

$$F^i A^{\otimes n} = \sum_{|J|=i,\ J \subset \{1,\ldots,n\}} y_J A^{\otimes n}.$$

This filtration is S_n-invariant. Let $F^i J_n = J_n \cap F^i A^{\otimes n}$ and $F^i R_n = \pi(F^i A^{\otimes n})$.

Let $R^{(i)}_n = gr^i R_n = F^i R_n / F^{i+1} R_n = F^i A^{\otimes n} / \left(F^i J_n + F^{i+1} A^{\otimes n}\right)$, then $R^{(n)}_n = 0$. We have

$$R^{(0)}_n = A^{\otimes n} / \left(F^0 J_n + F^1 A^{\otimes n}\right) = \mathbb{C}[x_1, \ldots, x_n]_{S_n} \simeq \mathbb{C}[S_n] \tag{7}$$

as an S_n-module ([C]).

Since the algebra R_n has the S_n-invariant filtration $\{F^i R_n\}_{0 \leq i \leq n}$, R_n is isomorphic to $\mathrm{gr}\, R_n = \bigoplus_{i=0}^{n-1} R^{(i)}_n$ by Lemma 3. For $1 \leq i \leq n-1$, we will determine the S_n-module structure of $R^{(i)}_n$ by using the result of Section 4. Since $F^i J_n + F^{i+1} A^{\otimes n} \subset I^n_{n-i,i+1}$, there is a homomorphism $\phi : R^{(i)}_n \to R^n_{n-i,i+1}$.

Definition 14. *Let $\mathcal{A}$ be a commutative ring and let $\mathcal{M}$ be an $\mathcal{A}$-module.*

1. *An element $a \in \mathcal{A}$ is called $\mathcal{M}$-regular if and only if for any $0 \neq x \in \mathcal{M}$ we have $ax \neq 0$.*
2. *A sequence $a_1, \ldots, a_n \in \mathcal{A}$ is called an $\mathcal{M}$-regular sequence if and only if for $j = 1, \ldots, n$, a_j is $(\mathcal{M}/\sum_{k=1}^{j-1} a_k \mathcal{M})$-regular.*

Lemma 15. *For any $n \in \mathbb{N}$, the sequence of the elementary symmetric polynomials $e_1, \ldots, e_n$ is the $\mathbb{C}[x_1, \ldots, x_n]$-regular sequence.*

Lemma 16. *Let $\mathcal{M}$ be a flat $\mathcal{A}$-module. If $f_1, \ldots, f_n \in \mathcal{A}$ is an $\mathcal{A}$-regular sequence, $f_1, \ldots, f_n$ is an $\mathcal{M}$-regular sequence.*

These lemmas are basic facts in the theory of commutative algebra.

Proposition 17. *For $1 \le i \le n-1$, the homomorphism $\phi : R_n^{(i)} \to R^n_{n-i,i+1}$ is injective.*

Proof. By the definition of ϕ, we only need to prove that $J_n + F^{i+1}A^{\otimes n} \supset F^i I^n_{n-i,i+1}$ for $1 \le i \le n-1$.

For $J \subset \{1, \dots, n\}$, let $\bar{J}$ be the complement of J in $\{1, \dots, n\}$. Fix an arbitrary element $P \in F^i I^n_{n-i,i+1}$. We can decompose P into two forms. First,

$$P = \sum_{|J|=i} P_J y_J + P' \tag{8}$$

where $P_J \in \mathbb{C}[x_j (j \in \bar{J})] \otimes \mathbb{C}[y_j (j \in J)]$ and $P' \in F^{i+1}A^{\otimes n}$. Second,

$$P = Q_1 e_1 + \cdots + Q_{n-i-1}e_{n-i-1} + \sum_{|I|=n-i} Q_I x_I + R_1 f_1 + \cdots + R_i f_i + \sum_{|J|=i+1} R_J y_J \tag{9}$$

where $Q_1, \dots, Q_{n-i-1}, Q_I\ R_1, \dots, R_i, R_J \in A^{\otimes n}$. Fix $J \subset \{1, \dots, n\}$, $|J| = i$. For $S \in A^{\otimes n}$, we denote by $\bar{S}$ the element of $\mathbb{C}[x_j (j \in \bar{J})] \otimes \mathbb{C}[y_j (j \in J)]$ obtained from the substitution $x_j = 0$ $(j \in J)$ and $y_j = 0$ $(j \in \bar{J})$. Set $x_j = 0$ for $j \in J$ and $y_j = 0$ for $j \in \bar{J}$ in (8) and (9), we have

$$\begin{aligned} P_J y_J = \bar{Q}_1 e_1(\{x_j\}_{j\in\bar{J}}) + \cdots + \bar{Q}_{n-i-1} e_{n-i-1}(\{x_j\}_{j\in\bar{J}}) + \bar{Q}_{\bar{J}} x_{\bar{J}} \\ + \bar{R}_1 f_1(\{y_j\}_{j\in J}) + \cdots + +\bar{R}_{i-1} f_{i-1}(\{y_j\}_{j\in J}) + \bar{R}_i y_J. \end{aligned}$$

Since $x_{\bar{J}} = e_{n-i}(\{x_j\}_{j\in\bar{J}})$ and $y_J = f_i(\{y_j\}_{j\in J})$, we have

$$(P_J - \bar{R}_i) f_i(\{y_j\}_{j\in J}) \in \left\langle \begin{matrix} e_1(\{x_j\}_{j\in\bar{J}}), \dots, e_{n-i}(\{x_j\}_{j\in\bar{J}}), \\ f_1(\{y_j\}_{j\in J}), \dots, f_{i-1}(\{y_j\}_{j\in J}) \end{matrix} \right\rangle_{\mathbb{C}[x_j(j\in\bar{J})]\otimes\mathbb{C}[y_j(j\in J)]} \tag{10}$$

in $\mathbb{C}[x_j(j \in \bar{J})] \otimes \mathbb{C}[y_j (j \in J)]$. Let $\mathcal{A} = \mathbb{C}[y_j (j \in J)]$ and let $\mathcal{M} = \mathbb{C}[x_j (j \in \bar{J})]_{S_{n-i}} \otimes \mathbb{C}[y_j (j \in J)]$. By Lemma 15, $f_1(\{y_j\}_{j\in J})$, …, $f_i(\{y_j\}_{j\in J})$ is the $\mathcal{A}$-regular sequence, and $\mathcal{M}$ is a flat $\mathcal{A}$-module. Hence, by Lemma 16, $f_1(\{y_j\}_{j\in J})$, …, $f_i(\{y_j\}_{j\in J})$ is an $\mathcal{M}$-regular sequence. Therefore, from (10), we have

$$P_J - \bar{R}_i \in \left\langle \begin{matrix} e_1(\{x_j\}_{j\in\bar{J}}), \dots, e_{n-i}(\{x_j\}_{j\in\bar{J}}), \\ f_1(\{y_j\}_{j\in J}), \dots, f_{i-1}(\{y_j\}_{j\in J}) \end{matrix} \right\rangle_{\mathbb{C}[x_j(j\in\bar{J})]\otimes\mathbb{C}[y_j(j\in J)]}.$$

By multiplying y_J we have

$$(P_J - R_i) y_J \in \left\langle \begin{matrix} e_1, \dots, e_{n-i}, \\ f_1, \dots, f_{i-1} \end{matrix} \right\rangle_{A^{\otimes n}} + F^{i+1}A^{\otimes n}.$$

Therefore, taking the summation for J $(|J| = i)$, we have

$$P - P' - R_i f_i \in \left\langle \begin{matrix} e_1, \dots, e_{n-i}, \\ f_1, \dots, f_{i-1} \end{matrix} \right\rangle_{A^{\otimes n}} + F^{i+1}A^{\otimes n} \subset J_n + F^{i+1}A^{\otimes n}.$$

Hence $P \in J_n + F^{i+1}A^{\otimes n}$. ◻

We have shown that $\phi : R_n^{(i)} \to R^n_{n-i,i+1}$ is an injective homomorphism. On the other hand, we have a surjective homomorphism $\psi : R^n_{n-i,i+1} \to R^n_{n-i,i}$ because $I^n_{n-i,i+1} \subset I^n_{n-i,i}$.

Proposition 18. *For $1 \le i \le n-1$, we have the following exact sequence of S_n-modules.*

$$0 \to R_n^{(i)} \xrightarrow{\phi} R^n_{n-i,i+1} \xrightarrow{\psi} R^n_{n-i,i} \to 0$$

Proof. First we have $\mathrm{Im}\phi \subset \mathrm{Ker}\psi$ because $F^i A^{\otimes n} \subset I^n_{n-i,i}$.

Next we show that $\mathrm{Im}\phi \supset \mathrm{Ker}\psi$. If $P \in \mathrm{Ker}\psi$, it belongs to the image of $I^n_{n-i,i}$ in $R^n_{n-i,i+1}$. Namely we have in $R^n_{n-i,i+1}$

$$\begin{aligned} P &= P_1 e_1 + \cdots + P_{n-i-1} e_{n-i-1} + \sum_{|I|=n-i} P_I x_I \\ &\qquad + Q_1 f_1 + \cdots + Q_{i-1} f_{i-1} + \sum_{|J|=i} Q_J y_J \\ &= \sum_{|J|=i} Q_J y_J \in F^i R^n_{n-i,i+1} = \phi(R_n^{(i)}). \end{aligned}$$

□

Corollary 19. *For $1 \le i \le n-1$, we have the following isomorphism of S_n-modules*:

$$R_n^{(i)} \simeq \mathrm{Ind}_{S_2}^{S_n} L_{(1,1)}.$$

Proof. By Proposition 11 and Proposition 13, $R^n_{n-i,i+1} \simeq \mathbb{C}[S_n]$ and $R^n_{n-i,i} \simeq \mathrm{Ind}_{S_2}^{S_n} L_{(2)}$, the claim of the corollary follows from the exact sequence of Proposition 18. □

Together with (7), we obtain the following theorem.

Theorem 20. *We have the following isomorphism of S_n-modules*:

$$R_n \simeq \mathbb{C}[S_n] \oplus (n-1)\,\mathrm{Ind}_{S_2}^{S_n} L_{(1,1)}.$$

6 The local Weyl module at a double point

In this section, we study the structure of the local Weyl module at the double point.

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be its triangular decomposition. Let M be an affine variety and let A be the coordinate ring of M. In [FL], Feigin and Loktev introduced the $(\mathfrak{g} \otimes A)$-module $W_M(\{0\}_\lambda)$ called the local Weyl module for a dominant integrable weight $\lambda \in \mathfrak{h}^*$. $W_M(\{0\}_\lambda)$ is the maximal $\mathfrak{g}$-integrable module with a cyclic vector v_0 such that:

$$(\mathfrak{n}_+ \otimes P)v_0 = 0, \qquad (h \otimes P)v_0 = \lambda(h)P(0)v_0 \quad (P \in A, h \in \mathfrak{h}).$$

Consider the case of $\mathfrak{g} = \mathfrak{sl}_{r+1}$. Let V_{r+1} be the vector representation of $\mathfrak{sl}_{r+1}$ and let ω_1 be the highest weight of V_{r+1}. For $\lambda = n\omega_1$, the following theorem is proved by Feigin and Loktev.

Theorem 21 ([FL]). *There is an isomorphism of $\mathfrak{sl}_{r+1}$-modules*:

$$W_M(\{0\}_{n\omega_1}) \simeq \left(V_{r+1}^{\otimes n} \otimes A_{S_n}^{\otimes n}\right)^{S_n}.$$

Thus, combining Theorem 20 and Theorem 21, we obtain the $\mathfrak{sl}_{r+1}$-module structure of $W_M(\{0\}_{n\omega_1})$ as follows.

Proposition 22. *For $n \in \mathbb{Z}_{\geq 0}$, we have the following isomorphism of $\mathfrak{sl}_{r+1}$-modules.*

$$W_M(\{0\}_{n\omega_1}) \simeq V_{r+1}^{\otimes n} \oplus (n-1)\left(V_{r+1}^{\otimes n-2} \otimes \wedge^2 V_{r+1}\right).$$

Proof. The following proof is essentially same as the first half of the proof of Theorem 10 in [FL].

By Theorem 21 and Theorem 20, we have

$$\begin{aligned}
W_M(\{0\}_{n\omega_1}) &\simeq \left(V_{r+1}^{\otimes n} \otimes A_{S_n}^{\otimes n}\right)^{S_n} \\
&\simeq \left(V_{r+1}^{\otimes n} \otimes (\mathbb{C}[S_n] \oplus (n-1)\operatorname{Ind}_{S_2}^{S_n} L_{(1,1)})\right)^{S_n} \\
&\simeq \left(V_{r+1}^{\otimes n} \otimes \mathbb{C}[S_n]\right)^{S_n} \oplus (n-1)\left(V_{r+1}^{\otimes n} \otimes \operatorname{Ind}_{S_2}^{S_n} L_{(1,1)}\right)^{S_n} \\
&\simeq \left(V_{r+1}^{\otimes n} \otimes \mathbb{C}[S_n]\right)^{S_n} \oplus (n-1)\left(V_{r+1}^{\otimes n} \otimes L_{(1^n)} \otimes \operatorname{Ind}_{S_2}^{S_n} L_{(2)}\right)^{S_n} \\
&\simeq V_{r+1}^{\otimes n} \oplus (n-1)\operatorname{Hom}_{S_n}(L_{(1^n)} \otimes (V_{r+1}^*)^{\otimes n}, \operatorname{Ind}_{S_2}^{S_n} L_{(2)}) \\
&\simeq V_{r+1}^{\otimes n} \oplus (n-1)\operatorname{Hom}_{S_2}(L_{(1^n)} \otimes (V_{r+1}^*)^{\otimes n}, L_{(2)}) \\
&\simeq V_{r+1}^{\otimes n} \oplus (n-1)V_{r+1}^{\otimes n-2} \otimes \wedge^2 V_{r+1}.
\end{aligned}$$

□

Corollary 23. *For $n \in \mathbb{Z}_{\geq 0}$, we have*

$$\dim W_M(\{0\}_{n\omega_1}) = (r+1)^{n-2}\left((r+1)^2 + \frac{(n-1)(r+1)r}{2}\right).$$

References

[C] C. Chevalley: Invariants of finite groups generated by reflections, Amer. J. Math. **78** (1955), 778-782.

[CP] V. Chari, A. Pressley: Weyl modules for Classical and Quantum Affine Algebras, Represent. Theory 5 (2001), 191-223, math.QA/0004174.

[FL] B. Feigin, S. Loktev: Multi-dimensional Weyl modules and symmetric functions, to appear in Comm. Math. Phys., math.QA/0212001.

[GP] A.M. Garsia, C. Procesi: On Certain Graded S_n-Modules and the q-Kostka Polynomials, Advances In Math. **94** (1992), 82-138.

[Ha] M. Haiman: Hilbert schemes, polygraphs and the Macdonald positivity conjecture, J. Amer. Math. Soc. **14** (2001), no. 4, 941-1006, math.AG/0001246.

[Hi] H.L. Hiller: Geometry of Coxeter Groups, Research Notes in Mathematics, No. 54, Pitman, Boston, 1982.

[W] H. Weyl: The Classical Groups. Their Invariants and Representations, Princeton University Press, Princeton, N.J., 1939.